

Deformations and Extensions of 3-Lie algebras

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Abstract

In this paper, we consider deformations of 3-Lie algebras via the cohomology theory. We prove that a 1-parameter infinitesimal deformation of a 3-Lie algebra \mathfrak{g} corresponds to a 1-cocycle of \mathfrak{g} with the coefficients in the adjoint representation. The notion of Nijenhuis operators for 3-Lie algebras is introduced to describe trivial deformations. We also study the abelian extension of 3-Lie algebras in details.

1 Introduction

In 1985, Filippov [8] introduced the concept of n -Lie algebra. An n -Lie algebra (also called Filippov algebra, Nambu algebra, Lie n -algebra, and so on) is a vector space \mathfrak{g} with an n -ary totally skew-symmetric linear map (n -bracket) from $\bigwedge^n \mathfrak{g}$ to \mathfrak{g} : $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_n]$ satisfying the n -Jacobi identity

$$[x_1, \dots, x_{n-1}, [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n [y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_n] \quad (1)$$

for all $x_i, y_i \in \mathfrak{g}$. When the n -ary linear map is not skew-symmetric, it is called n -Leibniz algebra or Leibniz n -algebra.

The study of n -Lie algebra is closely related to many fields in mathematics and mathematical physics. For example, L. Takhtajan in [15] developed the foundations of the theory of Nambu-Poisson manifolds. More recently, 3-Lie algebras is applied to the study of gauge symmetry and supersymmetry of multiple coincident M2-branes.

On the other hand, the algebraic theory of n -Lie algebras have been studied by many authors, see [4, 5, 7, 10, 12, 16]. Specially, the (co)homology theory for n -Lie algebra was introduced by L. Takhtajan in [16, 4] and by P. Gautheron in [5].

In this paper, we spell out the cohomology theory more precisely in the case $n = 3$, then we use it to study deformations of 3-Lie algebras. The notion of Nijenhuis operators for 3-Lie algebras is introduced and studied. We also verify that this kind of cohomology theory can be used to characterize abelian extension of 3-Lie algebras in the last section.

Throughout this paper, all 3-Lie algebras are assumed to be over an algebraically closed field of characteristic not equal to 2 and 3.

2 3-Lie algebras and cohomology

A 3-Lie algebra consists of a vector space \mathfrak{g} together with a linear map $[\cdot, \cdot, \cdot] : \bigwedge^3 \mathfrak{g} \rightarrow \mathfrak{g}$ such that the Jacobi identity

$$[x_1, x_2, [y_1, y_2, y_3]] = [[x_1, x_2, y_1], y_2, y_3] + [y_1, [x_1, x_2, y_2], y_3] + [y_1, y_2, [x_1, x_2, y_3]] \quad (2)$$

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holds for all $x_i, y_i \in \mathfrak{g}$.

Denote by $x = (x_1, x_2)$ and $\text{ad}(x)y_i = [x_1, x_2, y_i]$, then the above equality can be rewritten in the form

$$\text{ad}(x)[y_1, y_2, y_3] = [\text{ad}(x)y_1, y_2, y_3] + [y_1, \text{ad}(x)y_2, y_3] + [y_1, y_2, \text{ad}(x)y_3]. \quad (3)$$

Denoted by $\mathcal{L} := \bigwedge^2 \mathfrak{g}$, which is called fundamental set. The elements $x = (x_1, x_2) \in \bigwedge^2 \mathfrak{g}$ are called fundamental object. Define an operation on fundamental object by

$$x \circ y = ([x_1, x_2, y_1], y_2) + (y_1, [x_1, x_2, y_2]). \quad (4)$$

In [4], the authors proved that \mathcal{L} is a Leibniz algebra satisfying the following Leibniz rule

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z),$$

and

$$\text{ad}(x)\text{ad}(y)w - \text{ad}(y)\text{ad}(x)w = \text{ad}(x \circ y)w,$$

for all $x, y, z \in \mathcal{L}, w \in \mathfrak{g}$, i.e. $\text{ad} : \mathcal{L} \rightarrow \text{End}(\mathfrak{g})$ is a homomorphism of Leibniz algebras.

Recall that for a Leibniz algebra \mathcal{L} , a representation is a vector space V together with two bilinear maps

$$[\cdot, \cdot]_L : \mathcal{L} \times V \rightarrow V \quad \text{and} \quad [\cdot, \cdot]_R : V \times \mathcal{L} \rightarrow V$$

satisfying the following three axioms

- (LLM) $[x \circ y, m]_L = [x, [y, m]_L]_L - [y, [x, m]_L]_L,$
- (LML) $[m, x \circ y]_R = [[m, x]_R, y]_R + [x, [m, y]_R]_L,$
- (MLL) $[m, x \circ y]_R = [x, [m, y]_R]_L - [[x, m]_L, y]_R.$

By (LML) and (MLL) we also have

- (MMM) $[[m, x]_R, y]_R + [[x, m]_L, y]_R = 0.$

In fact, assume (LLM), one of (LML), (MLL), (MMM) can be derived from the other two.

Given a 3-Lie algebra \mathfrak{g} and a vector spaces V , define the maps

$$[\cdot, \cdot]_L : \mathcal{L} \otimes \text{Hom}(\mathfrak{g}, V) \rightarrow \text{Hom}(\mathfrak{g}, V) \quad \text{and} \quad [\cdot, \cdot]_R : \text{Hom}(\mathfrak{g}, V) \otimes \mathcal{L} \rightarrow \text{Hom}(\mathfrak{g}, V)$$

by

$$[(x_1, x_2), \phi]_L(x_3) = \rho(x_1, x_2)\phi(x_3) - \phi([x_1, x_2, x_3]), \quad (5)$$

$$\begin{aligned} [\phi, (x_1, x_2)]_R(x_3) &= \phi([x_1, x_2, x_3]) - \rho(x_1, x_2)\phi(x_3) - \rho(x_2, x_3)\phi(x_1) \\ &\quad - \rho(x_3, x_1)\phi(x_2), \end{aligned} \quad (6)$$

for all $\phi \in \text{Hom}(\mathfrak{g}, V), x_i \in \mathfrak{g}$, where ρ is a map from $\mathcal{L} = \bigwedge^2 \mathfrak{g}$ to $\text{End}(V)$.

Proposition 2.1. *Let \mathfrak{g} be a 3-Lie algebra. Then $\text{Hom}(\mathfrak{g}, V)$ equipped with the above two maps $[\cdot, \cdot]_L$ and $[\cdot, \cdot]_R$ is a representation of Leibniz algebra \mathcal{L} if and only if the following two conditions are satisfied, $\forall x_i, y_i \in \mathfrak{g}$,*

- (R1) $[\rho(x_1, x_2), \rho(y_1, y_2)] = \rho((x_1, x_2) \circ (y_1, y_2)),$
- (R2) $\rho(x_1, [y_1, y_2, y_3]) = \rho(y_2, y_3)\rho(x_1, y_1) + \rho(y_3, y_1)\rho(x_1, y_2) + \rho(y_1, y_2)\rho(x_1, y_3).$

Proof. For $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}, y_3 \in \mathfrak{g}$, first we compute the equality

$$[x \circ y, \phi]_L(y_3) = [x, [y, \phi]_L]_L(y_3) - [y, [x, \phi]_L]_L(y_3).$$

By definition, the left hand side is equal to

$$[x \circ y, \phi]_L(y_3) = \rho(x \circ y)\phi(y_3) - \phi(\text{ad}(x \circ y)y_3),$$

and the right hand side is equal to

$$\begin{aligned} & [x, [y, \phi]_L]_L(y_3) - [y, [x, \phi]_L]_L(y_3) \\ = & \rho(x)[y, \phi]_L(y_3) - [y, \phi]_L(\text{ad}(x)y_3) - \rho(y)[x, \phi]_L(y_3) + [x, \phi]_L(\text{ad}(y)y_3) \\ = & \rho(x)\rho(y)\phi(y_3) - \rho(x)\phi(\text{ad}(y)y_3) - \rho(y)\phi(\text{ad}(x)y_3) + \phi(\text{ad}(y)\text{ad}(x)y_3) \\ & - \rho(y)\rho(x)\phi(y_3) + \rho(y)\phi(\text{ad}(x)y_3) + \rho(x)\phi(\text{ad}(y)y_3) - \phi(\text{ad}(x)\text{ad}(y)y_3) \\ = & \rho(x)\rho(y)\phi(y_3) + \phi(\text{ad}(y)\text{ad}(x)y_3) - \rho(y)\rho(x)\phi(y_3) - \phi(\text{ad}(x)\text{ad}(y)y_3) \\ = & [\rho(x), \rho(y)]\phi(y_3) - \phi([\text{ad}(x), \text{ad}(y)]y_3). \end{aligned}$$

Since $\text{ad} : \mathcal{L} \rightarrow \text{End}(\mathfrak{g})$ is a homomorphism of Leibniz algebras, thus (LLM) is valid for $[\cdot, \cdot]_L$ if and only if (R1) is valid for ρ .

Next we compute the equality

$$[[\phi, x]_R, y]_R(y_3) + [[x, \phi]_L, y]_R(y_3) = 0.$$

By (5) and (6) we have

$$[(x_1, x_2), \phi]_L(w) + [\phi, (x_1, x_2)]_R(w) = \rho(x_1, w)\phi(x_2) - \rho(x_2, w)\phi(x_1),$$

thus

$$[(x_1, x_2), \phi]_L + [\phi, (x_1, x_2)]_R = \rho(x_1, \cdot)\phi(x_2) - \rho(x_2, \cdot)\phi(x_1),$$

where we denote $\rho(x_1, \cdot)\phi(x_2) : \mathfrak{g} \rightarrow V$ by $\rho(x_1, \cdot)\phi(x_2)(w) = \rho(x_1, w)\phi(x_2)$.

Now replace x_i by y_i and ϕ by $-(\rho(x_1, \cdot)\phi(x_2) + \rho(x_2, \cdot)\phi(x_1))$ in (6), then we have

$$\begin{aligned} & [[\phi, x]_R + [x, \phi]_L, y]_R(y_3) \\ = & (\rho(x_1, \cdot)\phi(x_2) - \rho(x_2, \cdot)\phi(x_1))([y_1, y_2, y_3]) \\ & - \rho(y_1, y_2)(\rho(x_1, \cdot)\phi(x_2) + \rho(x_2, \cdot)\phi(x_1))(y_3) \\ & - \rho(y_3, y_1)(\rho(x_1, \cdot)\phi(x_2) + \rho(x_2, \cdot)\phi(x_1))(y_2) \\ & - \rho(y_2, y_3)(\rho(x_1, \cdot)\phi(x_2) + \rho(x_2, \cdot)\phi(x_1))(y_1) \\ = & \rho(x_1, [y_1, y_2, y_3])\phi(x_2) - \rho(x_2, [y_1, y_2, y_3])\phi(x_1) \\ & - \rho(y_1, y_2)\rho(x_1, y_3)\phi(x_2) + \rho(y_1, y_2)\rho(x_2, y_3)\phi(x_1) \\ & - \rho(y_3, y_1)\rho(x_1, y_2)\phi(x_2) + \rho(y_1, y_3)\rho(x_2, y_2)\phi(x_1) \\ & - \rho(y_2, y_3)\rho(x_1, y_1)\phi(x_2) + \rho(y_2, y_3)\rho(x_2, y_1)\phi(x_1). \end{aligned}$$

thus (MMM) is valid for $[\cdot, \cdot]_L$ and $[\cdot, \cdot]_R$ if and only if (R2) is valid for ρ .

At last, we compute the equality

$$[\phi, x \circ y]_R(y_3) = [x, [\phi, y]_R]_L(y_3) - [[x, \phi]_L, y]_R(y_3).$$

By definition, the left hand side is equal to

$$\begin{aligned}
[\phi, x \circ y]_R(y_3) &= [\phi, ([x_1, x_2, y_1], y_2)]_R(y_3) + [\phi, (y_1, [x_1, x_2, y_2])]_R(y_3) \\
&= \phi([x_1, x_2, y_1], y_2, y_3) - \rho([x_1, x_2, y_1], y_2)\phi(y_3) \\
&\quad - \rho(y_3, [x_1, x_2, y_1])\phi(y_2) - \rho(y_2, y_3)\phi([x_1, x_2, y_1]) \\
&\quad + \phi([y_1, [x_1, x_2, y_2], y_3]) - \rho(y_1, [x_1, x_2, y_2])\phi(y_3) \\
&\quad - \rho(y_3, y_1)\phi([x_1, x_2, y_2]) - \rho([x_1, x_2, y_2], y_3)\phi(y_1),
\end{aligned}$$

and the right hand side is

$$\begin{aligned}
[x, [\phi, y]_R]_L(y_3) &= \rho(x_1, x_2)[\phi, (y_1, y_2)]_R(y_3) - [\phi, (y_1, y_2)]_R([x_1, x_2, y_3]) \\
&= \rho(x_1, x_2)\{\phi([y_1, y_2, y_3]) - \rho(y_1, y_2)\phi(y_3) \\
&\quad - \rho(y_3, y_1)\phi(y_2) - \rho(y_2, y_3)\phi(y_1)\} \\
&\quad - \{\phi([y_1, y_2, [x_1, x_2, y_3]]) - \rho(y_1, y_2)\phi([x_1, x_2, y_3]) \\
&\quad - \rho([x_1, x_2, y_3], y_1)\phi(y_2) - \rho(y_2, [x_1, x_2, y_3])\phi(y_1)\},
\end{aligned}$$

$$\begin{aligned}
[[x, \phi]_L, y]_R(y_3) &= [x, \phi]_L([y_1, y_2, y_3]) - \rho(y_1, y_2)[x, \phi]_L(y_3) \\
&\quad - \rho(y_3, y_1)[x, \phi]_L(y_2) - \rho(y_2, y_3)[x, \phi]_L(y_1) \\
&= \rho(x_1, x_2)\phi([y_1, y_2, y_3]) - \phi([x_1, x_2, [y_1, y_2, y_3]]) \\
&\quad - \rho(y_1, y_2)\{\rho(x_1, x_2)\phi(y_3) - \phi([x_1, x_2, y_3])\} \\
&\quad - \rho(y_3, y_1)\{\rho(x_1, x_2)\phi(y_2) - \phi([x_1, x_2, y_2])\} \\
&\quad - \rho(y_2, y_3)\{\rho(x_1, x_2)\phi(y_1) - \phi([x_1, x_2, y_1])\}.
\end{aligned}$$

Thus (MLL) is valid for $[\cdot, \cdot]_L$ if and only if (R1) and (R2) hold. \square

We remark that the necessity of the above Proposition was announced without proof for n -Leibniz algebras in [3].

Definition 2.2. Let \mathfrak{g} be a 3-Lie algebra and V be a vector space. Then (V, ρ) is called a representation of \mathfrak{g} or a \mathfrak{g} -module if and only if the conditions (R1) and (R2) in the above Proposition 2.1 are satisfied.

For example, given a 3-Lie algebra \mathfrak{g} , there is a natural **adjoint representation** on itself. The corresponding representation $\text{ad}(x_1, x_2)$ is given by

$$\text{ad}(x_1, x_2)x_3 = [x_1, x_2, x_3].$$

Now we define the generalized Chevalley-Eilenberg complex for a 3-Lie algebra \mathfrak{g} with coefficients in V by

$$C^n(\mathfrak{g}, V) := \text{Hom}\left(\bigwedge^{2n+1}\mathfrak{g}, V\right) \subseteq \text{Hom}\left(\left(\bigwedge^{2n}\mathfrak{g}\right) \otimes \mathfrak{g}, V\right) \cong \text{Hom}(\mathcal{L}^n, \text{Hom}(\mathfrak{g}, V))$$

and

$$d_{n-1} : C^{n-1}(\mathfrak{g}, V) \rightarrow C^n(\mathfrak{g}, V)$$

where

$$\begin{aligned}
& d_{n-1}\omega(x^1, x^2, \dots, x^n, w) \\
= & d_{n-1}\omega(x^1, x^2, \dots, x^n)(w) \\
= & \sum_{i=1}^{n-1} (-1)^{i+1} [x^i, \omega(x^1, \dots, \hat{x}^i, \dots, x^n)]_L(w) + (-1)^n [\omega(x^1, \dots, x^{n-1}), x^n]_R(w) \\
& + \sum_{1 \leq i < j \leq n} (-1)^{j+1} \omega(x^1, \dots, x^{i-1}, x^i \circ x^j, x^{i+1}, \hat{x}^j, \dots, x^n)(w),
\end{aligned}$$

for all $x^i \in \mathcal{L} = \bigwedge^2 \mathfrak{g}$, $w \in \mathfrak{g}$. In other words, we define the cohomology of a 3-Lie algebra \mathfrak{g} with coefficients in V to be the cohomology of Leibniz algebra \mathcal{L} with coefficients in $\text{Hom}(\mathfrak{g}, V)$. For more details of cohomology of Leibniz algebras, see [14].

Put $x^k = (x_{2k-1}, x_{2k})$, $w = x_{2n+1}$ and $[\cdot, \cdot]_L$, $[\cdot, \cdot]_R$ as in (5) and (6), then we get a coboundary operator $d_{n-1} : C^{n-1}(\mathfrak{g}, V) \rightarrow C^n(\mathfrak{g}, V)$ as the following:

$$\begin{aligned}
& d_{n-1}\omega(x_1, x_2, \dots, x_{2n+1}) := d_{n-1}\omega(x^1, x^2, \dots, x^n)(w) \\
= & \sum_{k=1}^{n-1} (-1)^{k+1} [x_{2k-1}, x_{2k}, \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, x_{2n})]_L(x_{2n+1}) \\
& + (-1)^n [\omega(x_1, x_2, \dots, x_{2n-2}), x_{2n-1}, x_{2n}]_R(x_{2n+1}) \\
& + \sum_{k=1}^n \sum_{j=2k+1}^{2n} (-1)^k \omega(x_1, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, [x_{2k-1}, x_{2k}, x_j], \dots, x_{2n})(x_{2n+1}) \\
= & \sum_{k=1}^{n-1} (-1)^{k+1} \{ \rho(x_{2k-1}, x_{2k}) \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, x_{2n+1}) \\
& - \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, [x_{2k-1}, x_{2k}, x_{2n+1}]) \} \\
& + (-1)^n \{ \omega(x_1, x_2, \dots, [x_{2n-1}, x_{2n}, x_{2n+1}]) - \rho(x_{2n-1}, x_{2n}) \omega(x_1, x_2, \dots, x_{2n+1}) \\
& - \rho(x_{2n+1}, x_{2n-1}) \omega(x_1, x_2, \dots, x_{2n-2}, x_{2n}) - \rho(x_{2n}, x_{2n+1}) \omega(x_1, x_2, \dots, x_{2n-1}) \} \\
& + \sum_{k=1}^n \sum_{j=2k+1}^{2n} (-1)^k \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, [x_{2k-1}, x_{2k}, x_j], \dots, x_{2n+1}) \\
= & (-1)^{n+1} \rho(x_{2n+1}, x_{2n-1}) \omega(x_1, x_2, \dots, x_{2n-2}, x_{2n}) \\
& + (-1)^{n+1} \rho(x_{2n}, x_{2n+1}) \omega(x_1, x_2, \dots, x_{2n-1}) \\
& + \sum_{k=1}^n (-1)^{k+1} \rho(x_{2k-1}, x_{2k}) \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, x_{2n+1}) \\
& + \sum_{k=1}^n \sum_{j=2k+1}^{2n+1} (-1)^k \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, [x_{2k-1}, x_{2k}, x_j], \dots, x_{2n+1}).
\end{aligned}$$

Theorem 2.3. *Let \mathfrak{g} be a 3-Lie algebra and (V, ρ) be a \mathfrak{g} -module. Then there exists a*

cochain complex $\{C(\mathfrak{g}, V) = \bigoplus_{n \geq 0} C^n(\mathfrak{g}, V), d\}$, where the coboundary operator is given by

$$\begin{aligned}
& d_{n-1}\omega(x_1, x_2, \dots, x_{2n+1}) \\
= & (-1)^{n+1}\rho(x_{2n+1}, x_{2n-1})\omega(x_1, x_2, \dots, x_{2n-2}, x_{2n}) \\
& + (-1)^{n+1}\rho(x_{2n}, x_{2n+1})\omega(x_1, x_2, \dots, x_{2n-1}) \\
& + \sum_{k=1}^n (-1)^{k+1}\rho(x_{2k-1}, x_{2k})\omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, x_{2n+1}) \\
& + \sum_{k=1}^n \sum_{j=2k+1}^{2n+1} (-1)^k \omega(x_1, x_2, \dots, \widehat{x_{2k-1}}, \widehat{x_{2k}}, \dots, [x_{2k-1}, x_{2k}, x_j], \dots, x_{2n+1}). \quad (7)
\end{aligned}$$

such that $d \circ d = 0$.

Definition 2.4. The quotient space $H^\bullet(\mathfrak{g}, V) = Z^\bullet(\mathfrak{g}, V)/B^\bullet(\mathfrak{g}, V)$, where $Z^\bullet(\mathfrak{g}, V) = \{\omega \in C^n(\mathfrak{g}, V) | d\omega = 0\}$ is the space of cocycles and $B^\bullet(\mathfrak{g}, V) = \{\omega = d\nu | \nu \in C^{n-1}(\mathfrak{g}, V)\}$ is the space of coboundaries, is called the cohomology group of a 3-Lie algebra \mathfrak{g} with coefficients in V .

According to the above definition, a 0-cochain is a map $\nu \in \text{Hom}(\mathfrak{g}, V)$, a 1-cochain is a map $\omega \in \text{Hom}(\bigwedge^3 \mathfrak{g}, V)$.

Definition 2.5. Let \mathfrak{g} be a 3-Lie algebra and (V, ρ) be a \mathfrak{g} -module. Then a map $\nu \in \text{Hom}(\mathfrak{g}, V)$ is called 0-cocycle if and only if $\forall x_1, x_2, x_3 \in \mathfrak{g}$,

$$\rho(x_1, x_2)\nu(x_3) + \rho(x_1, x_3)\nu(x_2) + \rho(x_2, x_3)\nu(x_1) - \nu([x_1, x_2, x_3]) = 0, \quad (8)$$

and a map $\omega \in \text{Hom}(\bigwedge^3 \mathfrak{g}, V)$ is called a 1-coboundary if there exists a map $\nu \in \text{Hom}(\mathfrak{g}, V)$ such that $\omega = d_0\nu$.

Definition 2.6. Let \mathfrak{g} be a 3-Lie algebra and (V, ρ) be a \mathfrak{g} -module. Then a map $\omega \in \text{Hom}(\bigwedge^3 \mathfrak{g}, V)$ is called 1-cocycle if and only if $\forall x_1, x_2, y_1, y_2, y_3 \in \mathfrak{g}$,

$$\begin{aligned}
& \omega(x_1, x_2, [y_1, y_2, y_3]) + \rho(x_1, x_2)\omega(y_1, y_2, y_3) \\
= & \omega([x_1, x_2, y_1], y_2, y_3) + \omega([x_1, x_2, y_2], y_3, y_1) + \omega(y_1, y_2, [x_1, x_2, y_3]) \\
& + \rho(y_2, y_3)\omega(x_1, x_2, y_1) + \rho(y_3, y_1)\omega(x_1, x_2, y_2) + \rho(y_1, y_2)\omega(x_1, x_2, y_3). \quad (9)
\end{aligned}$$

3 Infinitesimal Deformations

Let \mathfrak{g} be a 3-Lie algebra, and $\omega : \bigwedge^3 \mathfrak{g} \rightarrow \mathfrak{g}$ be a linear map. Consider a λ -parametrized family of linear operations:

$$[x_1, x_2, x_3]_\lambda \triangleq [x_1, x_2, x_3] + \lambda\omega(x_1, x_2, x_3).$$

If $[\cdot, \cdot, \cdot]_\lambda$ endow \mathfrak{g} with a 3-Lie algebra structure which is denoted by \mathfrak{g}_λ , then we say that ω generates a λ -parameter infinitesimal deformation of the 3-Lie algebra \mathfrak{g} .

Theorem 3.1. ω generates a λ -parameter infinitesimal deformation of the 3-Lie algebra \mathfrak{g} is equivalent to (i) ω itself defines a 3-Lie algebras structure on \mathfrak{g} and (ii) ω is a 1-cocycle of \mathfrak{g} with the coefficients in the adjoint representation.

Proof. For the equality

$$\begin{aligned} & [x_1, x_2, [y_1, y_2, y_3]_\lambda]_\lambda \\ = & [[x_1, x_2, y_1]_\lambda, y_2, y_3]_\lambda + [y_1, [x_1, x_2, y_2]_\lambda, y_3]_\lambda + [y_1, y_2, [x_1, x_2, y_3]_\lambda]_\lambda, \end{aligned}$$

the left hand side is equal to

$$\begin{aligned} & [x_1, x_2, [y_1, y_2, y_3] + \lambda\omega(y_1, y_2, y_3)]_\lambda \\ = & [x_1, x_2, [y_1, y_2, y_3]] + \lambda\omega(x_1, x_2, [y_1, y_2, y_3]) \\ & + [x_1, x_2, \lambda\omega(y_1, y_2, y_3)] + \lambda\omega(x_1, x_2, \lambda\omega(y_1, y_2, y_3)) \\ = & [x_1, x_2, [y_1, y_2, y_3]] + \lambda\{\omega(x_1, x_2, [y_1, y_2, y_3]) + [x_1, x_2, \omega(y_1, y_2, y_3)]\} \\ & + \lambda^2\omega(x_1, x_2, \omega(y_1, y_2, y_3)), \end{aligned}$$

and the right hand side is equal to

$$\begin{aligned} & [[x_1, x_2, y_1] + \lambda\omega(x_1, x_2, y_1), y_2, y_3]_\lambda + [y_1, [x_1, x_2, y_2] + \lambda\omega(x_1, x_2, y_2), y_3]_\lambda \\ & + [y_1, y_2, [x_1, x_2, y_3] + \lambda\omega(x_1, x_2, y_3)]_\lambda \\ = & [[x_1, x_2, y_1], y_2, y_3] + [y_1, [x_1, x_2, y_2], y_3] + [y_1, y_2, [x_1, x_2, y_3]] \\ & + \lambda\{\omega([x_1, x_2, y_1], y_2, y_3) + [\omega(x_1, x_2, y_1), y_2, y_3] \\ & + \omega(y_1, [x_1, x_2, y_2], y_3) + [y_1, \omega(x_1, x_2, y_2), y_3] \\ & + \omega(y_1, y_2, [x_1, x_2, y_3]) + [y_1, y_2, \omega(x_1, x_2, y_3)]\} \\ & + \lambda^2\{\omega(\omega(x_1, x_2, y_1), y_2, y_3) + \omega(y_1, \omega(x_1, x_2, y_2), y_3) + \omega(y_1, y_2, \omega(x_1, x_2, y_3))\}. \end{aligned}$$

Thus we have

$$\begin{aligned} & \omega(x_1, x_2, [y_1, y_2, y_3]) + [x_1, x_2, \omega(y_1, y_2, y_3)] \\ = & \omega([x_1, x_2, y_1], y_2, y_3) + \omega(y_1, [x_1, x_2, y_2], y_3) + \omega(y_1, y_2, [x_1, x_2, y_3]) \\ & + [\omega(x_1, x_2, y_1), y_2, y_3] + [y_1, \omega(x_1, x_2, y_2), y_3] + [y_1, y_2, \omega(x_1, x_2, y_3)], \end{aligned} \quad (10)$$

$$\begin{aligned} & \omega(x_1, x_2, \omega(y_1, y_2, y_3)) \\ = & \omega(\omega(x_1, x_2, y_1), y_2, y_3) + \omega(y_1, \omega(x_1, x_2, y_2), y_3) + \omega(y_1, y_2, \omega(x_1, x_2, y_3)). \end{aligned} \quad (11)$$

Therefore ω defines a 3-Lie algebra structure on \mathfrak{g} and ω is a 1-cocycle of \mathfrak{g} with the coefficients in the adjoint representation. \square

4 Nijenhuis operators

In this section, we introduce the notion of Nijenhuis operators for 3-Lie algebras, which is analogy to the case of ordinary Lie algebras in [6, 11] and of associative algebras in [2]. This kind of operator gives trivial deformation.

A deformation is said to be **trivial** if there exists a linear map $N : \mathfrak{g} \rightarrow \mathfrak{g}$ such that for $T_\lambda = \text{id} + \lambda N$: $\mathfrak{g}_\lambda \rightarrow \mathfrak{g}$ there holds

$$T_\lambda([x_1, x_2, x_3]_\lambda) = [T_\lambda x_1, T_\lambda x_2, T_\lambda x_3]. \quad (12)$$

By definition we have

$$\begin{aligned} T_\lambda([x_1, x_2, x_3]_\lambda) &= [x_1, x_2, x_3] + \lambda\omega(x_1, x_2, x_3) + \lambda N([x_1, x_2, x_3] + \lambda\omega(x_1, x_2, x_3)) \\ &= [x_1, x_2, x_3] + \lambda(\omega(x_1, x_2, x_3) + N[x_1, x_2, x_3]) + \lambda^2 N\omega(x_1, x_2, x_3), \end{aligned}$$

and

$$\begin{aligned}
[T_\lambda x_1, T_\lambda x_2, T_\lambda x_3] &= [x_1 + \lambda N x_1, x_2 + \lambda N x_2, x_3 + \lambda N x_3] \\
&= [x_1, x_2, x_3] + \lambda([N x_1, x_2, x_3] + [x_1, N x_2, x_3] + [x_1, x_2, N x_3]) \\
&\quad + \lambda^2([N x_1, N x_2, x_3] + [N x_1, x_2, N x_3] + [x_1, N x_2, N x_3]) \\
&\quad + \lambda^3[N x_1, N x_2, N x_3].
\end{aligned}$$

Thus we have

$$\omega(x_1, x_2, x_3) = [N x_1, x_2, x_3] + [x_1, N x_2, x_3] + [x_1, x_2, N x_3] - N[x_1, x_2, x_3], \quad (13)$$

$$N\omega(x_1, x_2, x_3) = [N x_1, N x_2, x_3] + [N x_1, x_2, N x_3] + [x_1, N x_2, N x_3], \quad (14)$$

$$0 = [N x_1, N x_2, N x_3]. \quad (15)$$

From the cohomology theory discussed in section 2, (13) can be represented in terms of 1-coboundary as $\omega = d_0 N$. Moreover, it follows from (13) and (14) that N must satisfy the following condition

$$\begin{aligned}
N^2[x_1, x_2, x_3] &= N[N x_1, x_2, x_3] + N[x_1, N x_2, x_3] + N[x_1, x_2, N x_3] \\
&\quad - ([N x_1, N x_2, x_3] + [N x_1, x_2, N x_3] + [x_1, N x_2, N x_3]). \quad (16)
\end{aligned}$$

In the following, we denote by $\omega(x_1, x_2, x_3) = [x_1, x_2, x_3]_N$, then (16) is equivalent to

$$N[x_1, x_2, x_3]_N = [N x_1, N x_2, x_3] + [N x_1, x_2, N x_3] + [x_1, N x_2, N x_3]. \quad (17)$$

Definition 4.1. A linear operator $N : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a Nijenhuis operator if and only if (17) and (15) hold.

We have seen that any trivial deformation produces a Nijenhuis operator. Conversely, any Nijenhuis operator gives a trivial deformation as the following theorem shows.

Theorem 4.2. Let N be a Nijenhuis operator for \mathfrak{g} . Then a deformation of \mathfrak{g} can be obtained by putting

$$\omega(x_1, x_2, x_3) = [N x_1, x_2, x_3] + [x_1, N x_2, x_3] + [x_1, x_2, N x_3] - N[x_1, x_2, x_3].$$

Furthermore, this deformation is a trivial one.

Proof. We have known that $\omega = d_0 N$ and $d_1 \omega = d_1 d_0 N = 0$, therefore ω is a 1-cocycle of \mathfrak{g} with the coefficients in the adjoint representation. Now we check the Jacobi identity (2) hold for ω . Denote by

$$\begin{aligned}
J(x_1, x_2, y_1, y_2, y_3) &= [x_1, x_2, [y_1, y_2, y_3]] - [[x_1, x_2, y_1], y_2, y_3] \\
&\quad - [y_1, [x_1, x_2, y_2], y_3] - [y_1, y_2, [x_1, x_2, y_3]], \\
J^\omega(x_1, x_2, y_1, y_2, y_3) &= \omega(x_1, x_2, \omega(y_1, y_2, y_3)) - \omega(\omega(x_1, x_2, y_1), y_2, y_3) \\
&\quad - \omega(y_1, \omega(x_1, x_2, y_2), y_3) - \omega(y_1, y_2, \omega(x_1, x_2, y_3)).
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
&J^\omega(x_1, x_2, y_1, y_2, y_3) \\
&= J(N x_1, N x_2, y_1, y_2, y_3) + N^2 J(x_1, x_2, y_1, y_2, y_3) \\
&\quad + [x_1, x_2, [N y_1, N y_2, y_3] + [N y_1, y_2, N y_3] + [y_1, N y_2, N y_3] - N\omega(y_1, y_2, y_3)] \\
&\quad - ([y_1, y_2, [N x_1, N x_2, y_3] + [N x_1, x_2, N y_3] + [x_1, N x_2, N y_3] - N\omega(x_1, x_2, y_3)]) \\
&\quad - ([y_2, y_3, [N x_1, N x_2, y_1] + [N x_1, x_2, N y_1] + [x_1, N x_2, N y_1] - N\omega(x_1, x_2, y_1)]) \\
&\quad - ([y_3, y_1, [N x_1, N x_2, y_2] + [N x_1, x_2, N y_2] + [x_1, N x_2, N y_2] - N\omega(x_1, x_2, y_2)]).
\end{aligned}$$

Therefore we have $J^\omega = 0$ by the Leibniz rule of \mathfrak{g} and Nijenhuis operator condition. \square

Note that in the proof of above Theorem we have not used condition (14). But this condition is important to us since only in this case the k 's power of a Nijenhuis operator is also a Nijenhuis operator.

Lemma 4.3. *Let N be a Nijenhuis operator. Then for any $k > 0$, we have*

$$[x_1, x_2, x_3]_{N^{k+1}} = ([x_1, x_2, x_3]_{N^k})_N. \quad (18)$$

Proof. First we check that (18) is valid for $k = 1$.

$$\begin{aligned} ([x_1, x_2, x_3]_N)_N &= [Nx_1, x_2, x_3]_N + [x_1, Nx_2, x_3]_N + [x_1, x_2, Nx_3]_N - N[x_1, x_2, x_3]_N \\ &= [N^2x_1, x_2, x_3] + [Nx_1, Nx_2, x_3] + [Nx_1, x_2, Nx_3] - N[Nx_1, x_2, x_3] \\ &\quad + [Nx_1, Nx_2, x_3] + [x_1, N^2x_2, x_3] + [x_1, Nx_2, Nx_3] - N[x_1, Nx_2, x_3] \\ &\quad + [Nx_1, x_2, Nx_3] + [x_1, Nx_2, Nx_3] + [x_1, x_2, N^2x_3] - N[x_1, x_2, Nx_3] \\ &\quad - N[Nx_1, x_2, x_3] - N[x_1, Nx_2, x_3] - N[x_1, x_2, Nx_3] + N^2[x_1, x_2, x_3] \\ &= [N^2x_1, x_2, x_3] + [x_1, N^2x_2, x_3] + [x_1, x_2, N^2x_3] - N^2[x_1, x_2, x_3] \\ &= [x_1, x_2, x_3]_{N^2}, \end{aligned}$$

where we have used (16) in the third equality.

Second, assume that

$$[x_1, x_2, x_3]_{N^k} = ([x_1, x_2, x_3]_{N^{k-1}})_N. \quad (19)$$

We compute

$$\begin{aligned} &([x_1, x_2, x_3]_{N^k})_N \\ &= [N^kx_1, x_2, x_3]_N + [x_1, N^kx_2, x_3]_N + [x_1, x_2, N^kx_3]_N - N^k[x_1, x_2, x_3]_N \\ &= [N^{k+1}x_1, x_2, x_3] + [N^kx_1, Nx_2, x_3] + [N^kx_1, x_2, Nx_3] - N[N^kx_1, x_2, x_3] \\ &\quad + [Nx_1, N^kx_2, x_3] + [x_1, N^{k+1}x_2, x_3] + [x_1, N^kx_2, Nx_3] - N[x_1, N^kx_2, x_3] \\ &\quad + [Nx_1, x_2, N^kx_3] + [x_1, Nx_2, N^kx_3] + [x_1, x_2, N^{k+1}x_3] - N[x_1, x_2, N^kx_3] \\ &\quad - N^k[x_1, x_2, x_3]_N. \end{aligned}$$

By (15) and (17) we have

$$\begin{aligned} &[N^kx_1, Nx_2, x_3] + [N^kx_1, x_2, Nx_3] \\ &= [N^kx_1, Nx_2, x_3] + [N^kx_1, x_2, Nx_3] + [N^{k-1}x_1, Nx_2, Nx_3] \\ &= [NN^{k-1}x_1, Nx_2, x_3] + [NN^{k-1}x_1, x_2, Nx_3] + [N^{k-1}x_1, Nx_2, Nx_3] \\ &= N[N^{k-1}x_1, x_2, x_3]_N \end{aligned}$$

Similarly we have

$$[Nx_1, N^kx_2, x_3] + [x_1, N^kx_2, Nx_3] = N[x_1, N^{k-1}x_2, x_3]_N,$$

and

$$[Nx_1, x_2, N^kx_3] + [x_1, Nx_2, N^kx_3] = N[x_1, x_2, N^{k-1}x_3]_N.$$

These three items together with the last item in $([x_1, x_2, x_3]_{N^k})_N$ is equal to

$$\begin{aligned} &N[N^{k-1}x_1, x_2, x_3]_N + N[x_1, N^{k-1}x_2, x_3]_N + N[x_1, x_2, N^{k-1}x_3]_N - N^k[x_1, x_2, x_3]_N \\ &= N([x_1, x_2, x_3]_{N^{k-1}})_N \\ &= N([x_1, x_2, x_3]_{N^k}) \quad \text{by assumption (19)}. \end{aligned}$$

Thus $([x_1, x_2, x_3]_{N^k})_N$ is equal to

$$\begin{aligned}
&= [N^{k+1}x_1, x_2, x_3] + [x_1, N^{k+1}x_2, x_3] + [x_1, x_2, N^{k+1}x_3] \\
&\quad - N[N^kx_1, x_2, x_3] - N[x_1, N^kx_2, x_3] - N[x_1, x_2, N^kx_3] \\
&\quad + N([x_1, x_2, x_3]_{N^k}) \\
&= [N^{k+1}x_1, x_2, x_3] + [x_1, N^{k+1}x_2, x_3] + [x_1, x_2, N^{k+1}x_3] - N^{k+1}[x_1, x_2, x_3] \\
&= [x_1, x_2, x_3]_{N^{k+1}}.
\end{aligned}$$

and by introduction the Lemma holds. \square

By the above Lemma, we have

$$\begin{aligned}
([x_1, x_2, x_3]_{N^k})_{N^r} &= (([x_1, x_2, x_3]_{N^k})_N)_{N^{r-1}} = ([x_1, x_2, x_3]_{N^{k+1}})_{N^{r-1}} \\
&= ([x_1, x_2, x_3]_{N^{k+2}})_{N^{r-2}} = \cdots = [x_1, x_2, x_3]_{N^{k+r}}.
\end{aligned}$$

Lemma 4.4. *Let N be a Nijenhuis operator, then for any $k, r > 0$, we have*

$$[x_1, x_2, x_3]_{N^{k+r}} = ([x_1, x_2, x_3]_{N^k})_{N^r}. \quad (20)$$

Proposition 4.5. *Let N be a Nijenhuis operator. Then for any $k > 0$, N^k is also a Nijenhuis operator.*

Proof. We prove by induction. The Proposition is valid for $k = 1$. Assume

$$N^k[x_1, x_2, x_3]_{N^k} = [N^kx_1, N^kx_2, x_3] + [N^kx_1, x_2, N^kx_3] + [x_1, N^kx_2, N^kx_3],$$

then we have

$$\begin{aligned}
&N^{k+1}[x_1, x_2, x_3]_{N^{k+1}} \\
&= N^k N([x_1, x_2, x_3]_{N^k})_N \\
&= N^k([N^kx_1, N^kx_2, x_3]_{N^k} + [N^kx_1, x_2, N^kx_3]_{N^k} + [x_1, N^kx_2, N^kx_3]_{N^k}) \\
&= [N^{k+1}x_1, N^{k+1}x_2, x_3] + [N^{k+1}x_1, x_2, N^{k+1}x_3] + [N^kx_1, N^{k+1}x_2, N^kx_3] \\
&\quad + [N^{k+1}x_1, N^kx_2, N^kx_3] + [N^{k+1}x_1, x_2, N^{k+1}x_3] + [N^kx_1, N^{k+1}x_2, N^{k+1}x_3] \\
&\quad + [N^kx_1, N^{k+1}x_2, N^kx_3] + [N^kx_1, x_2, N^{k+1}x_3] + [x_1, N^{k+1}x_2, N^{k+1}x_3] \\
&= [N^{k+1}x_1, N^{k+1}x_2, x_3] + [N^{k+1}x_1, x_2, N^{k+1}x_3] + [x_1, N^{k+1}x_2, N^{k+1}x_3] \\
&\quad + [N^{k+1}x_1, N^kx_2, N^kx_3] + [N^kx_1, N^{k+1}x_2, N^kx_3] + [N^{k+1}x_1, N^kx_2, N^kx_3] \\
&\quad + [N^kx_1, N^{k+1}x_2, N^{k+1}x_3] + [N^kx_1, N^kx_2, N^{k+1}x_3] + [N^kx_1, N^kx_2, N^kx_3].
\end{aligned}$$

The items in the last two line are zero by (15), thus the Proposition is valid for $k + 1$. \square

Two Nijenhuis operator N_1 and N_2 are said to be compatible if $N_1 + N_2$ is also a Nijenhuis operator.

Proposition 4.6. *Let N_1 and N_2 be two Nijenhuis operators. Then they are compatible if and only if*

$$\begin{aligned}
&N_1[x_1, x_2, x_3]_{N_2} + N_2[x_1, x_2, x_3]_{N_1} \\
&= [N_2x_1, N_1x_2, x_3] + [N_2x_1, x_2, N_1x_3] + [x_1, N_2x_2, N_1x_3] \\
&\quad + [N_1x_1, N_2x_2, x_3] + [N_1x_1, x_2, N_2x_3] + [x_1, N_1x_2, N_2x_3],
\end{aligned} \quad (21)$$

and

$$\begin{aligned}
&[N_1x_1, N_1x_2, N_2x_3] + [N_1x_1, N_2x_2, N_2x_3] \\
&+ [N_2x_1, N_1x_2, N_1x_3] + [N_2x_1, N_2x_2, N_1x_3] = 0,
\end{aligned} \quad (22)$$

Proof. By definition, $N_1 + N_2$ is a Nijenhuis operator if and only if

$$\begin{aligned} & (N_1 + N_2)([x_1, x_2, x_3]_{N_1 + N_2}) \\ = & [(N_1 + N_2)x_1, (N_1 + N_2)x_2, x_3] + [(N_1 + N_2)x_1, x_2, (N_1 + N_2)x_3] \\ & + [x_1, (N_1 + N_2)x_2, (N_1 + N_2)x_3], \end{aligned}$$

and

$$[(N_1 + N_2)x_1, (N_1 + N_2)x_2, (N_1 + N_2)x_3] = 0.$$

Now it is easy to see the above condition is equivalent to (21) and (22). \square

Lemma 4.7. *Let N be a Nijenhuis operator. For any $j, k > 0$, we have*

$$\begin{aligned} & N^j[x_1, x_2, x_3]_{N^k} + N^k[x_1, x_2, x_3]_{N^j} \\ = & [N^k x_1, N^j x_2, x_3] + [N^k x_1, x_2, N^j x_3] + [x_1, N^k x_2, N^j x_3] \\ & + [N^j x_1, N^k x_2, x_3] + [N^j x_1, x_2, N^k x_3] + [x_1, N^j x_2, N^k x_3]. \end{aligned} \quad (23)$$

Proof. If $j > k$, then by Proposition 4.5 and Lemma 4.4 we have

$$\begin{aligned} & N^j[x_1, x_2, x_3]_{N^k} + N^k[x_1, x_2, x_3]_{N^j} \\ = & N^{j-k}(N^k[x_1, x_2, x_3]_{N^k}) + N^k([x_1, x_2, x_3]_{N^{j-k}})_{N^k} \\ = & N^{j-k}([N^k x_1, N^k x_2, x_3] + [N^k x_1, x_2, N^k x_3] + [x_1, N^k x_2, N^k x_3]) \\ & + [N^k x_1, N^k x_2, x_3]_{N^{j-k}} + [N^k x_1, x_2, N^k x_3]_{N^{j-k}} + [x_1, N^k x_2, N^k x_3]_{N^{j-k}} \\ = & N^{j-k}([N^k x_1, N^k x_2, x_3] + [N^k x_1, x_2, N^k x_3] + [x_1, N^k x_2, N^k x_3]) \\ & + [N^j x_1, N^k x_2, x_3] + [N^k x_1, N^j x_2, x_3] + [N^k x_1, N^k x_2, N^{j-k} x_3] - N^{j-k}[N^k x_1, N^k x_2, x_3] \\ & + [N^j x_1, x_2, N^k x_3] + [N^k x_1, N^{j-k} x_2, N^k x_3] + [N^k x_1, x_2, N^j x_3] - N^{j-k}[N^k x_1, x_2, N^k x_3] \\ & + [N^{j-k} x_1, N^k x_2, N^k x_3] + [x_1, N^j x_2, N^k x_3] + [x_1, N^k x_2, N^j x_3] - N^{j-k}[x_1, N^k x_2, N^k x_3] \\ = & [N^k x_1, N^j x_2, x_3] + [N^k x_1, x_2, N^j x_3] + [x_1, N^k x_2, N^j x_3] \\ & + [N^j x_1, N^k x_2, x_3] + [N^j x_1, x_2, N^k x_3] + [x_1, N^j x_2, N^k x_3]. \end{aligned}$$

The case of $j < k$ can be proved similarly and the case of $j = k$ is by Proposition 4.5. \square

Let $N_1 = N^j$ and $N_2 = N^k$, then condition (22) in Proposition 4.6 is satisfied by (15) and condition (21) is satisfied by Lemma 4.7, thus we get

Theorem 4.8. *Let N be a Nijenhuis operator. Then N^j and N^k are compatible for any $j, k > 0$.*

It is easy to see that if N is a Nijenhuis operator, then cN is also a Nijenhuis operator, where c is any constant. Now by Proposition 4.6 and Lemma 4.7 we have

Theorem 4.9. *Let N be a Nijenhuis operator. Then all linear combinations of N^k are compatible.*

Corollary 4.10. *Let N be a Nijenhuis operator. Then for any polynomial $P(X) = \sum_{i=1}^n c_i X^i$, the operator $P(N)$ is also a Nijenhuis operator.*

5 Abelian Extensions of 3-Lie algebras

In this section, we study abelian extensions of 3-Lie algebras. We show that associated to any abelian extension, there is a representation and a 1-cocycle. Furthermore, abelian extensions can be classified by the first cohomology group.

Definition 5.1. Let $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$, $(V, [\cdot, \cdot, \cdot]_V)$, $(\hat{\mathfrak{g}}, [\cdot, \cdot, \cdot]_{\hat{\mathfrak{g}}})$ be 3-Lie algebras and $i : V \rightarrow \hat{\mathfrak{g}}$, $p : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$ be homomorphisms. The following sequence of 3-Lie algebras is a short exact sequence if $\text{Im}(i) = \text{Ker}(p)$, $\text{Ker}(i) = 0$ and $\text{Im}(p) = \mathfrak{g}$.

$$0 \longrightarrow V \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 0 \quad (24)$$

In this case, we call $\hat{\mathfrak{g}}$ an extension of \mathfrak{g} by V , and denote it by $E_{\hat{\mathfrak{g}}}$. It is called an abelian extension if V is abelian ideal of $\hat{\mathfrak{g}}$, i.e. $[\cdot, u, v]_{\hat{\mathfrak{g}}} = 0$, $\forall u, v \in V$.

A section $\sigma : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ of $p : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$ consists of linear maps $\sigma : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ such that $p \circ \sigma = \text{id}_{\mathfrak{g}}$.

Definition 5.2. Two extensions of 3-Lie algebras $E_{\hat{\mathfrak{g}}} : 0 \rightarrow V \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \rightarrow 0$ and $E_{\tilde{\mathfrak{g}}} : 0 \rightarrow V \xrightarrow{j} \tilde{\mathfrak{g}} \xrightarrow{q} \mathfrak{g} \rightarrow 0$ are equivalent, if there exists a 3-Lie algebras homomorphism $F : \hat{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$ such that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow F & & \downarrow \text{id} \\ 0 & \longrightarrow & V & \xrightarrow{j} & \tilde{\mathfrak{g}} & \xrightarrow{q} & \mathfrak{g} \longrightarrow 0 \end{array} \quad (25)$$

The set of equivalent classes of extensions of \mathfrak{g} by V is denoted by $\text{Ext}(\mathfrak{g}, V)$.

Let $\hat{\mathfrak{g}}$ be an abelian extension of \mathfrak{g} by V , and $\sigma : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ be a section. Denote by

$$\sigma(x) = \sigma(x_1, x_2) = (\sigma x_1, \sigma x_2),$$

and define $\rho : \wedge^2 \mathfrak{g} \rightarrow \text{End}(V)$ by

$$\rho(x)(u) = \rho(x_1, x_2)(u) \triangleq [\sigma(x_1), \sigma(x_2), u]_{\hat{\mathfrak{g}}} = \text{ad}(\sigma(x))u, \quad (26)$$

for all $x = (x_1, x_2) \in \wedge^2 \mathfrak{g}$, $u \in V$.

Lemma 5.3. *With the above notations, ρ is a representation of \mathfrak{g} on V and does not depend on the choice of the section σ . Moreover, equivalent abelian extensions give the same representation of \mathfrak{g} on V .*

Proof. First, we show that ρ is independent of the choice of σ . In fact, if we choose another section $\sigma' : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$, then

$$p(\sigma(x_i) - \sigma'(x_i)) = x_i - x_i = 0 \implies \sigma(x_i) - \sigma'(x_i) \in V \implies \sigma'(x_i) = \sigma(x_i) + u$$

for some $u \in V$.

Since we have $[\cdot, u, v]_{\hat{\mathfrak{g}}} = 0$ for all $u, v \in V$, which implies that

$$\begin{aligned} [\sigma'(x_1), \sigma'(x_2), w]_{\hat{\mathfrak{g}}} &= [\sigma(x_1) + u, \sigma(x_2) + v, w]_{\hat{\mathfrak{g}}} \\ &= [\sigma(x_1), \sigma(x_2) + v, w]_{\hat{\mathfrak{g}}} + [u, \sigma(x_2) + v, w]_{\hat{\mathfrak{g}}} \\ &= [\sigma(x_1), \sigma(x_2), w]_{\hat{\mathfrak{g}}} + [\sigma(x_1), v, w]_{\hat{\mathfrak{g}}} \\ &= [\sigma(x_1), \sigma(x_2), w]_{\hat{\mathfrak{g}}}, \end{aligned}$$

thus ρ is independent on the choice of σ .

Second, we show that ρ is a representation of \mathfrak{g} on V .

By the equality

$$\begin{aligned} & [\sigma x_1, \sigma x_2, [\sigma y_1, \sigma y_2, u]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}} \\ = & [[\sigma x_1, \sigma x_2, \sigma y_1]_{\hat{\mathfrak{g}}}, \sigma y_2, u]_{\hat{\mathfrak{g}}} + [\sigma y_1, [\sigma x_1, \sigma x_2, \sigma y_2]_{\hat{\mathfrak{g}}}, u]_{\hat{\mathfrak{g}}} + [\sigma y_1, \sigma y_2, [\sigma x_1, \sigma x_2, u]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}}, \end{aligned}$$

we have

$$\rho(x_1, x_2)\rho(y_1, y_2)u = \rho((x_1, x_2) \circ (y_1, y_2))u + \rho(y_1, y_2)\rho(x_1, x_2)u,$$

where we use the fact

$$\sigma([x_1, x_2, y_1]_{\hat{\mathfrak{g}}}) - [\sigma x_1, \sigma x_2, \sigma y_1]_{\hat{\mathfrak{g}}} \in V \cong \text{Ker}(p),$$

and that V is abelian ideal of $\hat{\mathfrak{g}}$,

$$[\sigma([x_1, x_2, y_1]_{\hat{\mathfrak{g}}}) - [\sigma x_1, \sigma x_2, \sigma y_1]_{\hat{\mathfrak{g}}}, \sigma y_2, u]_{\hat{\mathfrak{g}}} = 0,$$

thus we get the condition (R1).

Similarly, by the equality

$$\begin{aligned} & [u, \sigma x_1, [\sigma y_1, \sigma y_2, \sigma y_3]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}} \\ = & [[u, \sigma x_1, \sigma y_1]_{\hat{\mathfrak{g}}}, \sigma y_2, \sigma y_3]_{\hat{\mathfrak{g}}} + [\sigma y_1, [u, \sigma x_1, \sigma y_2]_{\hat{\mathfrak{g}}}, \sigma y_3]_{\hat{\mathfrak{g}}} + [\sigma y_1, \sigma y_2, [u, \sigma x_1, \sigma y_3]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}}, \end{aligned}$$

we have

$$\rho(x_1, [y_1, y_2, y_3])u = \rho(y_2, y_3)\rho(x_1, y_1)u + \rho(y_3, y_1)\rho(x_1, y_2)u + \rho(y_1, y_2)\rho(x_1, y_3)u.$$

thus we get the condition (R2). Therefore we see that ρ is a representation of \mathfrak{g} on V .

At last, suppose that $E_{\hat{\mathfrak{g}}}$ and $E_{\tilde{\mathfrak{g}}}$ are equivalent abelian extensions, and $F : \hat{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$ is the 3-Lie algebra homomorphism satisfying $F \circ i = j$, $q \circ F = p$. Choose linear sections σ and σ' of p and q , we get $qF\sigma(x_i) = p\sigma(x_i) = x_i = q\sigma'(x_i)$, then $F\sigma(x_i) - \sigma'(x_i) \in \text{Ker}(q) \cong V$. Thus, we have

$$[\sigma(x_1), \sigma(x_2), u]_{\hat{\mathfrak{g}}} = [F\sigma(x_1), F\sigma(x_2), u]_{\tilde{\mathfrak{g}}} = [\sigma'(x_1), \sigma'(x_2), u]_{\tilde{\mathfrak{g}}}.$$

Therefore, equivalent abelian extensions give the same ρ . The proof is finished. \square

Let $\sigma : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ be a section of the abelian extension. Define the following map:

$$\omega(x_1, x_2, x_3) \triangleq [\sigma(x_1), \sigma(x_2), \sigma(x_3)]_{\hat{\mathfrak{g}}} - \sigma([x_1, x_2, x_3]_{\mathfrak{g}}), \quad (27)$$

for all $x_1, x_2, x_3 \in \mathfrak{g}$.

Lemma 5.4. *Let $0 \rightarrow V \rightarrow \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow 0$ be an abelian extension of \mathfrak{g} by V . Then ω defined by (27) is a 1-cocycle of \mathfrak{g} with coefficients in V , where the representation ρ is given by (26).*

Proof. By the equality

$$\begin{aligned} & [\sigma x_1, \sigma x_2, [\sigma y_1, \sigma y_2, \sigma y_3]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}} \\ = & [[\sigma x_1, \sigma x_2, \sigma y_1]_{\hat{\mathfrak{g}}}, \sigma y_2, \sigma y_3]_{\hat{\mathfrak{g}}} + [\sigma y_1, [\sigma x_1, \sigma x_2, \sigma y_2]_{\hat{\mathfrak{g}}}, \sigma y_3]_{\hat{\mathfrak{g}}} \\ & + [\sigma y_1, \sigma y_2, [\sigma x_1, \sigma x_2, \sigma y_3]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}}, \end{aligned}$$

we get that the left hand side is equal to

$$\begin{aligned}
& [\sigma x_1, \sigma x_2, [\sigma y_1, \sigma y_2, \sigma y_3]_{\hat{\mathfrak{g}}}]_{\hat{\mathfrak{g}}} \\
&= [\sigma x_1, \sigma x_2, \omega(y_1, y_2, y_3) + \sigma([y_1, y_2, y_3]_{\mathfrak{g}})]_{\hat{\mathfrak{g}}} \\
&= \rho(x_1, x_2)\omega(y_1, y_2, y_3) + [\sigma x_1, \sigma x_2, \sigma([y_1, y_2, y_3]_{\mathfrak{g}})]_{\hat{\mathfrak{g}}} \\
&= \rho(x_1, x_2)\omega(y_1, y_2, y_3) + \omega(x_1, x_2, [y_1, y_2, y_3]) + \sigma([x_1, x_2, [y_1, y_2, y_3]]).
\end{aligned}$$

Similarly, the right hand side is equal to

$$\begin{aligned}
& \rho(y_2, y_3)\omega(x_1, x_2, y_1) + \omega([x_1, x_2, y_1], y_2, y_3) + \sigma([x_1, x_2, y_1], y_2, y_3) \\
&+ \rho(y_3, y_1)\omega(x_1, x_2, y_2) + \omega([y_1, [x_1, x_2, y_2], y_3]) + \sigma([y_1, [x_1, x_2, y_2], y_3]) \\
&+ \rho(y_1, y_2)\omega(x_1, x_2, y_3) + \omega([y_1, y_2, [x_1, x_2, y_3]]) + \sigma([y_1, y_2, [x_1, x_2, y_3]]).
\end{aligned}$$

Thus we have

$$\begin{aligned}
& \omega(x_1, x_2, [y_1, y_2, y_3]) + \rho(x_1, x_2)\omega(y_1, y_2, y_3) \\
&= \omega([x_1, x_2, y_1], y_2, y_3) + \omega([x_1, x_2, y_2], y_3, y_1) + \omega(y_1, y_2, [x_1, x_2, y_3]) \\
&+ \rho(y_2, y_3)\omega(x_1, x_2, y_1) + \rho(y_3, y_1)\omega(x_1, x_2, y_2) + \rho(y_1, y_2)\omega(x_1, x_2, y_3).
\end{aligned}$$

This is exactly the 1-cocycle condition in Definition 2.6. \square

Now we can transfer the 3-Lie algebra structure on $\hat{\mathfrak{g}}$ to the 3-Lie algebra structure on $\mathfrak{g} \oplus V$ using the 1-cocycle given above. Lemma 5.5 below can be found in [1, 13] but we present the proof for the sake of completeness.

Lemma 5.5. *Let \mathfrak{g} be a 3-Lie algebra, (V, ρ) be an \mathfrak{g} -module and $\omega : \bigwedge^3 \mathfrak{g} \rightarrow V$ is a 1-cocycle. Then $\mathfrak{g} \oplus V$ is a 3-Lie algebra under the following multiplication:*

$$\begin{aligned}
& [x_1 + u_1, x_2 + u_2, x_3 + u_3]_{\omega} \\
&= [x_1, x_2, x_3] + \omega(x_1, x_2, x_3) + \rho(x_1, x_2)(u_3) + \rho(x_2, x_3)(u_1) + \rho(x_3, x_1)(u_2),
\end{aligned}$$

where $x_1, x_2, x_3 \in \mathfrak{g}$ and $u_1, u_2, u_3 \in V$. This 3-Lie algebra is denoted by $\mathfrak{g} \oplus_{\omega} V$.

Proof. Now it suffices to verify the fundamental identity.

$$\begin{aligned}
& [x_1 + u_1, x_2 + u_2, [y_1 + v_1, y_2 + v_2, y_3 + v_3]_{\omega}]_{\omega} \\
&= [x_1 + u_1, x_2 + u_2, [y_1, y_2, y_3] + \omega(y_1, y_2, y_3) + \rho(y_1, y_2)(v_3) + \rho(y_3, y_1)(v_2) + \rho(y_2, y_3)(v_1)] \\
&= [x_1, x_2, [y_1, y_2, y_3]] + \omega(x_1, x_2, [y_1, y_2, y_3]) + \rho([y_1, y_2, y_3], x_1)(u_2) + \rho(x_2, [y_1, y_2, y_3])(u_1) \\
&+ \rho(x_1, x_2)\{\omega(y_1, y_2, y_3) + \rho(y_1, y_2)(v_3) + \rho(y_3, y_1)(v_2) + \rho(y_2, y_3)(v_1)\}, \\
& [[x_1 + u_1, x_2 + u_2, y_1 + v_1]_{\omega}, y_2 + v_2, y_3 + v_3]_{\omega} \\
&= [x_1, x_2, y_1] + \omega(x_1, x_2, y_1) + \rho(x_1, x_2)(v_1) + \rho(y_1, x_1)(u_2) + \rho(x_2, y_1)(u_1), y_2 + v_2, y_3 + v_3] \\
&= [[x_1, x_2, y_1], y_2, y_3] + \omega([x_1, x_2, y_1], y_2, y_3) + \rho([x_1, x_2, y_1], y_2)(v_3) + \rho(y_3, [x_1, x_2, y_1])(v_2) \\
&+ \rho(y_2, y_3)\{\omega(x_1, x_2, y_1) + \rho(x_1, x_2)(v_1) + \rho(y_1, x_1)(u_2) + \rho(x_2, y_1)(u_1)\}, \\
& [y_1 + v_1, [x_1 + u_1, x_2 + u_2, y_2 + v_2]_{\omega}, y_3 + v_3]_{\omega} \\
&= [y_1 + v_1, [x_1, x_2, y_2] + \omega(x_1, x_2, y_2) + \rho(x_1, x_2)(v_2) + \rho(y_2, x_1)(u_2) + \rho(x_2, y_2)(u_1)], y_3 + v_3] \\
&= [y_1, [x_1, x_2, y_2], y_3] + \omega(y_1, [x_1, x_2, y_2], y_3) + \rho(y_1, [x_1, x_2, y_2])(v_3) + \rho([x_1, x_2, y_2], y_3)(v_1) \\
&+ \rho(y_3, y_1)\{\omega(x_1, x_2, y_2) + \rho(x_1, x_2)(v_2) + \rho(y_2, x_1)(u_2) + \rho(x_2, y_2)(u_1)\},
\end{aligned}$$

$$\begin{aligned}
& [y_1 + v_1, y_2 + v_2, [x_1 + u_1, x_2 + u_2, y_3 + v_3]_\omega]_\omega \\
&= [y_1 + v_1, y_2 + v_2, [x_1, x_2, y_3] + \omega(x_1, x_2, y_3) + \rho(x_1, x_2)(v_3) + \rho(y_3, x_1)(u_2) + \rho(x_2, y_3)(u_1)] \\
&= [y_1, y_2, [x_1, x_2, y_3]] + \omega(y_1, y_2, [x_1, x_2, y_3]) + \rho([x_1, x_2, y_3], y_1)(v_2) + \rho(y_2, [x_1, x_2, y_3])(v_1) \\
&\quad + \rho(y_1, y_2)\{\omega(x_1, x_2, y_3) + \rho(x_1, x_2)(v_3) + \rho(y_3, x_1)(u_2) + \rho(x_2, y_3)(u_1)\},
\end{aligned}$$

It follows that

$$\begin{aligned}
& [x_1 + u_1, x_2 + u_2, [y_1 + v_1, y_2 + v_2, y_3 + v_3]_\omega]_\omega \\
&= [[x_1 + u_1, x_2 + u_2, y_1 + v_1]_\omega, y_2 + v_2, y_3 + v_3]_\omega \\
&\quad + [y_1 + v_1, [x_1 + u_1, x_2 + u_2, y_2 + v_2]_\omega, y_3 + v_3]_\omega \\
&\quad + [y_1 + v_1, y_2 + v_2, [x_1 + u_1, x_2 + u_2, y_3 + v_3]_\omega]_\omega
\end{aligned}$$

since ρ is a representation and ω is a 1-cocycle. \square

Lemma 5.6. *Two abelian extensions of 3-Lie algebras $0 \rightarrow V \rightarrow \mathfrak{g} \oplus_\omega V \rightarrow \mathfrak{g} \rightarrow 0$ and $0 \rightarrow V \rightarrow \mathfrak{g} \oplus_{\omega'} V \rightarrow \mathfrak{g} \rightarrow 0$ are equivalent if and only if ω and ω' are in the same cohomology class.*

Proof. Let $F : \mathfrak{g} \oplus_\omega V \rightarrow \mathfrak{g} \oplus_{\omega'} V$ be the corresponding homomorphism, then

$$F[x_1, x_2, x_3]_\omega = [F(x_1), F(x_2), F(x_3)]_{\omega'}. \quad (28)$$

Since F is an equivalence of extensions, there exist $\nu : \mathfrak{g} \rightarrow V$ such that

$$F(x_i + u) = x_i + \nu(x_i) + u, \quad i = 1, 2, 3.$$

The left hand side of (28) is equal to

$$\begin{aligned}
& F_1([x_1, x_2, x_3] + \omega(x_1, x_2, x_3)) \\
&= [x_1, x_2, x_3] + \omega(x_1, x_2, x_3) + \nu([x_1, x_2, x_3]),
\end{aligned}$$

and the right hand side of (28) is equal to

$$\begin{aligned}
& [x_1 + \nu(x_1), x_2 + \nu(x_2), x_3 + \nu(x_3)]_{\omega'} \\
&= [x_1, x_2, x_3] + \omega'(x_1, x_2, x_3) \\
&\quad + \rho(x_1, x_2)\nu(x_3) + \rho(x_1, x_3)\nu(x_2) + \rho(x_2, x_3)\nu(x_1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
(\omega - \omega')(x_1, x_2, x_3) &= \rho(x_1, x_2)\nu(x_3) + \rho(x_1, x_3)\nu(x_2) + \rho(x_2, x_3)\nu(x_1) \\
&\quad - \nu([x_1, x_2, x_3]),
\end{aligned} \quad (29)$$

that is $\omega - \omega' = d\nu$. Therefore ω and ω' are in the same cohomology class. \square

Theorem 5.7. *Let \mathfrak{g} be a 3-Lie algebra and (V, ρ) be a \mathfrak{g} -module. Then there is a one-to-one correspondence between equivalence classes of abelian extensions of the 3-Lie algebra \mathfrak{g} by V and the first cohomology group $\mathbf{H}^1(\mathfrak{g}, V)$.*

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